required from the actual antenna, assuming equal power densities in the given direction. Thus,

$$g(\theta) = \frac{4\pi r^2 S_r(\theta)}{\bar{\mathcal{P}}_f} \tag{2-130}$$

For $L \leq \lambda$, the maximum gain of a dipole antenna occurs at $\theta = \pi/2$. From Eqs. (2-126) and (2-128), we have

$$g\left(\frac{\pi}{2}\right) = \frac{\eta|I_m|^2\left(1 - \cos\frac{kL}{2}\right)^2}{\pi\bar{\mathcal{P}}_f} = \frac{\eta\left(1 - \cos\frac{kL}{2}\right)^2}{\pi R_r} \tag{2-131}$$

In the limit $kL \to 0$, we have $g(\pi/2) = 1.5$; so the maximum gain of a short dipole is 1.5. For a half-wave dipole, we can use Fig. 2.24 and calculate a maximum gain of 1.64. Similarly, for a full-wave dipole, the maximum gain is 2.41.

The *input impedance* of an antenna is the impedance seen by the source, that is, the ratio of the complex terminal voltage to the complex terminal current. A knowledge of the reactive power, which cannot be obtained from radiation zone fields, is needed to evaluate the input reactance. The input resistance accounts for the radiated power (and dissipated power if losses are present). We define the input resistance of a loss-free antenna as

$$R_i = \frac{\bar{\mathcal{P}}_f}{|I_i|^2} \tag{2-132}$$

where $\bar{\mathcal{P}}_f$ is the power radiated and I_i is the input current. If losses are present, a "loss resistance" must be added to Eq. (2-132) to obtain the input resistance. For the dipole antenna,

$$I_i = I_m \sin\frac{kL}{2}$$

and the input resistance is

$$R_i = \frac{R_r}{\sin^2[k(L/2)]} \tag{2-133}$$

In the limit as kL is made small, we find

$$R_i = \frac{\eta(kL)^2}{24\pi} \qquad L \ll \lambda \tag{2-134}$$

The short dipole therefore has a very small input resistance. For example, if $L = \lambda/10$, the input resistance is about 2 ohms. For the half-wavelength dipole, we use Fig. 2-24 and Eq. (2-133) and find

$$R_i = R_r = 73.1 \text{ ohms} \qquad L = \frac{\lambda}{2} \tag{2-135}$$

For the full-wavelength dipole, Eq. (2-133) shows $R_i = \infty$. This incorrect result is due to our initial choice of current, which has a null at the source. The input resistance of the full-wavelength dipole is actually large, but not infinite, and depends markedly on the wire diameter (see Fig. 7-13).

2-11. On Waves in General. A complex function of coordinates representing an instantaneous function according to Eq. (1-40) is called a *wave function.* A wave function ψ, which may be either a scalar field or the component of a vector field, may be expressed as

$$\psi = A(x,y,z)e^{j\Phi(x,y,z)} \tag{2-136}$$

where A and Φ are real. The corresponding instantaneous function is

$$\sqrt{2}\, A(x,y,z) \cos\left[\omega t + \Phi(x,y,z)\right] \tag{2-137}$$

The *magnitude* A of the complex function is the rms amplitude of the instantaneous function. The *phase* Φ of the complex function is the initial phase of the instantaneous function. Surfaces over which the phase is constant (instantaneous function vibrates in phase) are called *equiphase surfaces.* These are defined by

$$\Phi(x,y,z) = \text{constant} \tag{2-138}$$

Waves are called *plane, cylindrical,* or *spherical* according as their equiphase surfaces are planes, cylinders, or spheres. Waves are called *uniform* when the amplitude A is constant over the equiphase surfaces. Perpendiculars to the equiphase surfaces are called *wave normals.* These are, of course, in the direction of $\nabla\Phi$ and are the curves along which the phase changes most rapidly.

The rate at which the phase decreases in some direction is called the *phase constant* in that direction. (The term phase constant is used even though it is not, in general, a constant.) For example, the phase constants in the cartesian coordinate directions are

$$\beta_x = -\frac{\partial\Phi}{\partial x} \qquad \beta_y = -\frac{\partial\Phi}{\partial y} \qquad \beta_z = -\frac{\partial\Phi}{\partial z} \tag{2-139}$$

These may be considered as components of a *vector phase constant* defined by

$$\boldsymbol{\beta} = -\nabla\Phi \tag{2-140}$$

The maximum phase constant is therefore along the wave normal and is of magnitude $|\nabla\Phi|$.

The *instantaneous phase* of a wave is the argument of the cosine function of Eq. (2-137). A *surface of constant phase* is defined as

$$\omega t + \Phi(x,y,z) = \text{constant} \tag{2-141}$$

that is, the instantaneous phase is constant. At any instant, the surfaces of constant phase coincide with the equiphase surfaces. As time increases, Φ must decrease to maintain the constancy of Eq. (2-141), and the surfaces of constant phase move in space. For any increment $d\mathbf{s}$ the change in Φ is

$$\nabla\Phi \cdot d\mathbf{s} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

To keep the instantaneous phase constant for an incremental increase in time, we must have

$$\omega\,dt + \nabla\Phi \cdot d\mathbf{s} = 0$$

That is, the total differential of Eq. (2-141) must vanish. The *phase velocity* of a wave in a given direction is defined as the velocity of surfaces of constant phase in that direction. For example, the phase velocities along cartesian coordinates are

$$\begin{aligned} v_x &= -\frac{\omega}{\partial\Phi/\partial x} = \frac{\omega}{\beta_x} \\ v_y &= -\frac{\omega}{\partial\Phi/\partial y} = \frac{\omega}{\beta_y} \\ v_z &= -\frac{\omega}{\partial\Phi/\partial z} = \frac{\omega}{\beta_z} \end{aligned} \tag{2-142}$$

The phase velocity along a wave normal ($d\mathbf{s}$ in the direction of $-\nabla\Phi$) is

$$v_p = -\frac{\omega}{|\nabla\Phi|} = \frac{\omega}{\beta} \tag{2-143}$$

which is the *smallest* phase velocity for the wave. Phase velocity is *not* a vector quantity.

We can also express the wave function, Eq. (2-136), as

$$\psi = e^{\Theta(x,y,z)} \tag{2-144}$$

where Θ is a complex function whose imaginary part is the phase Φ. A *vector propagation constant* can be defined in terms of the rate of change of Θ as

$$\boldsymbol{\gamma} = -\nabla\Theta = \boldsymbol{\alpha} + j\boldsymbol{\beta} \tag{2-145}$$

where $\boldsymbol{\beta}$ is the phase constant of Eq. (2-140) and $\boldsymbol{\alpha}$ is the *vector attenuation constant.* The components of $\boldsymbol{\alpha}$ are the logarithmic rates of change of the magnitude of ψ in the various directions.

In the electromagnetic field, ratios of components of **E** to components of **H** are called *wave impedances.* The direction of a wave impedance is defined according to the right-hand "cross-product" rule of component **E** rotated into component **H**. For example,

$$\frac{E_x}{H_y} = Z_{xy}{}^+ = Z_z \tag{2-146}$$

is a wave impedance in the $+z$ direction, while

$$-\frac{E_x}{H_y} = Z_{xy}{}^- = Z_{-z} \tag{2-147}$$

is a wave impedance in the $-z$ direction. The wave impedance in the $+z$ direction involving E_y and H_x is

$$\frac{-E_y}{H_x} = Z_{yx}{}^+ = -Z_{yx}{}^- \tag{2-148}$$

The Poynting vector can be expressed in terms of wave impedances. For example, the z component is

$$\begin{aligned} S_z = (\mathbf{E} \times \mathbf{H}^*)_z &= E_xH_y^* - E_yH_x^* \\ &= Z_{xy}{}^+|H_y|^2 + Z_{yx}{}^+|H_x|^2 \end{aligned} \tag{2-149}$$

The concept of wave impedance is most useful when the wave impedances are constant over equiphase surfaces.

Let us illustrate the various concepts by specializing them to the uniform plane wave. Consider the x-polarized z-traveling wave in lossy matter,

$$\begin{aligned} E_x &= E_0e^{-k''z}e^{-jk'z} \\ H_y &= \frac{E_0}{\eta}e^{-k''z}e^{-jk'z} \end{aligned}$$

The amplitude of E_x is $E_0e^{-k''z}$ and its phase is $-k'z$. Equiphase surfaces are defined by $-k'z$ = constant, or, since k' is constant, by z = constant. These are planes; so the wave is a plane wave. The amplitude of E_x is constant over each equiphase surface; so the wave is uniform. The wave normals all point in the z direction. The cartesian components of the phase constant are $\beta_x = \beta_y = 0$, $\beta_z = k'$; so the vector phase constant is $\boldsymbol{\beta} = \mathbf{u}_zk'$. The phase velocity in the direction of the wave normals is $v_p = \omega/k'$. The cartesian components of the attenuation constant are $\alpha_x = \alpha_y = 0$, $\alpha_z = k''$; so the vector attenuation constant is $\boldsymbol{\alpha} = \mathbf{u}_zk''$. The vector propagation constant is

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} + j\boldsymbol{\beta} = \mathbf{u}_z(k'' + jk') = \mathbf{u}_zjk$$

The wave impedance in the z direction is $Z_z = Z_{xy}{}^+ = E_x/H_y = \eta$. Note that the various parameters specialized to the uniform plane traveling wave are all intrinsic parameters. This is, by definition, the meaning of the word "intrinsic."